Tunnelling in 'massless' quantum mechanics

1992 J. Phys. A: Math. Gen. 25 L439

(http://iopscience.iop.org/0305-4470/25/8/010)

J. Phys. A: Math. Gen. 25 (1992) L439–L442. Printed in the UK

LETTER TO THE EDITOR

Tunnelling in 'massless' quantum mechanics

Ilya V Krive†, Boris A Malomed‡ and Alexander S Rozhavsky†

† Institute for Low Temperature Physics and Engineering, Ukrainian Academy of Sciences, 47 Lenin Avenue, 310164 Kharkov, Ukraine
‡ P P Shirshov Institute for Oceanology, Academy of Sciences, 23 Krasikov Street, Moscow, Russia, and Department of Applied Mathematics, School of Mathematical Sciences, Tel-Aviv University, Ramat Ariv, 69978, Israel

Received 11 October 1991

Abstract. The problem of tunnelling in 'topological' (massless) quantum mechanics with an anharmonic potential is considered. A simple method to calculate the imaginary part of action for the family of potentials, $U(x, y) = (xy)^n$, is proposed. The tunnelling action is determined by the pole contribution only (on the complex-time plane), in a contrast to the ordinary (massive) quantum mechanics, in which tunnelling action is always determined by the branch point contribution.

It is well known that some non-trivial aspects of quantum field theories can be adequately modelled by quantum mechanical systems with a finite number of degrees of freedom. An interesting problem in the quantum field theory is the vacuum–vacuum tunnelling, which is treated usually as the instanton contribution to the classical Euclidean action. This problem was thoroughly analysed for the models with quadratic kinetic term in Lagrangians (see e.g. [1]).

In the past few years new topological field theories in odd spacetime dimensions have attracted a great deal of interest. These models contain linear time derivative kinetic terms, and their dynamical properties differ essentially from those of traditional theories. At present, these so-called Chern–Simons models are of particular interest not only in quantum field theory but in solid state physics also, because of their relevance to the problem of fractional statistics [2, 3]. It is thus interesting to consider the tunnelling problem for Chern–Simons-type Lagrangians.

The simplest model which mimics Chern–Simons theory is the 'topological' massless quantum mechanics which was first introduced in [4]. It is the aim of our letter to analyse tunnelling in a model with the Lagrangian:

$$L = \tfrac{1}{2}(x\dot{y} - \dot{x}y) - U(x, y). \tag{1}$$

One may regard (1) as the Lagrangian of a rotator in the (x, y) plane in the presence of an external potential $U(x, y)$, or as a massless charged particle moving on a plane under the action of the same potential and magnetic field, directed normal to the plane.

The quantum mechanical formalism utilizes summing up the Feynman amplitudes, $\exp(iS)$, where S is the classical action corresponding to an arbitrary path lying on a fixed-energy surface and connecting given initial and final points. In the traditional models with the quadratic kinetic energy, the trajectories passing through classically forbidden regions, where the potential energy U exceeds the total energy E, give rise

0305-4470/92/080439+04$04.50

to the imaginary part of the corresponding action. This is the commonly known description of the under-barrier transition.

In model (1), the total energy coincides with the potential U, so that this standard philosophy is irrelevant. However, we will demonstrate that the forbidden trajectories and an exponentially small probability for the under-barrier transition exist as well in models of type (1).

In contrast to [4] where the harmonic rotationally symmetric potential, $U(x,y)\sim x^2+2y^2$, was considered, we choose the form of potential energy which allows tunnelling. For simplicity, in subsequent analysis we select the particular potentials:

$$U=(xy)^n \tag{2}$$

with an integer n. On the phase plane (x, y), the surfaces of equal energy are the hyperbolas, $xy=\text{constant}$, see figure 1. A sub-barrier trajectory connects two points belonging to the different branches of the hyperbola corresponding to the same energy (figure 1).

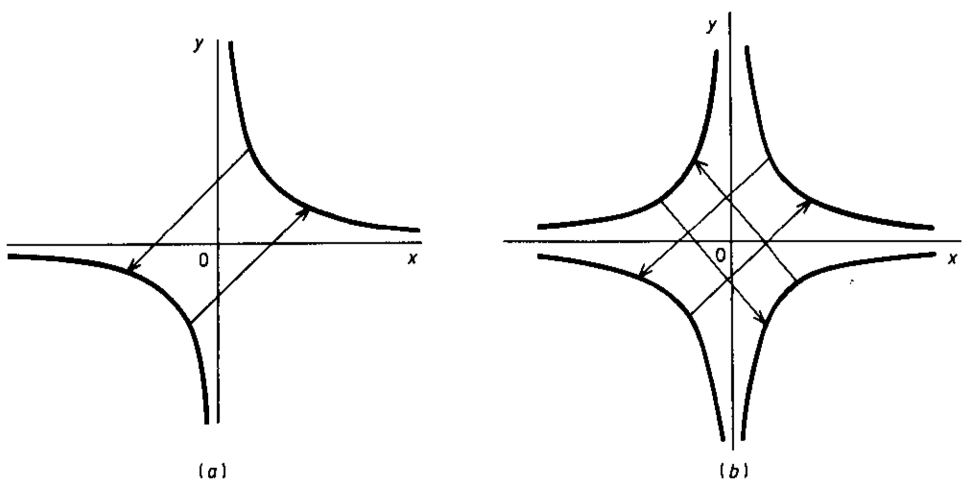

Figure 1. The equal-energy hyperbolas on the phase plane of the model (1) and (2): (a) n is odd; (b) n is even. The arrows indicate possible paths of the tunnelling across the classically forbidden regions.

Following the general prescription of quantum mechanics, we describe the under-barrier paths by formal solution of the classical motion equations corresponding to the Lagrangian (1):

$$\dot{x}=-\frac{\partial U}{\partial y} \tag{3a}$$

$$\dot{y}=\frac{\partial U}{\partial x} \tag{3b}$$

with complex time t. For the quadratic Lagrangians, the under-barrier motion is accomplished in purely imaginary time, the coordinate taking purely real values. In the present case, the sub-barrier path goes along some contour on the complex plane of t, which maps into a contour on the complex plane of the coordinate with real initial and final points. The action, S, along the pass is given by the integral:

$$S=\int L\,\mathrm{d}t$$

$$=-\int(\dot{x}y+U(x,y))\,\mathrm{d}t+\tfrac{1}{2}(x_\mathrm{f}y_\mathrm{f}-x_\mathrm{i}y_\mathrm{i})$$

$$=\int\left[\frac{U(x,y)}{\partial V/\partial y}-y\right]\mathrm{d}x. \tag{4}$$

In (4), $(x_\mathrm{i}, y_\mathrm{i})$ and $(x_\mathrm{f}, y_\mathrm{f})$ stand for the coordinates of the initial and final points of the complex classical under-barrier path (figure 1), the last term being produced when integrating by parts the term $\frac{1}{2}x\dot{y}$ in the Lagrangian (1). Equation (3b) has been used to transform the integrand. To perform the integration, one should insert into (4) y as a function $y(x)$ determined by the energy conservation

$$U(\dot{x},y)=U_\mathrm{i}=U(x_\mathrm{i},y_\mathrm{i}). \tag{5}$$

Let us now proceed to the particular potentials (2), designating the constant value of energy $U_\mathrm{i}=u^n$. In this case, (5) yields

$$y(x)=U/x \tag{6}$$

and substitution of (6) and (2) into (4) brings us to the integral the eventual form of which is:

$$S=-\frac{n-1}{n}u\int_{x_\mathrm{i}}^{x_\mathrm{f}}\frac{\mathrm{d}x}{x}. \tag{7}$$

The path of integration going along the real axis gives rise to a divergence in the integral (7), provided the signs of x_i and x_f are opposite. This reflects the impossibility of finding a usual (real-valued) trajectory connecting the two branches of the equal-energy hyperbola lying at $x>0$ and at $x<0$ (figure 1). Following the general prescription, we use a contour bypassing the pole of the integrand at $x=0$, which is equivalent to rewriting the integral as follows:

$$S=-\frac{n-1}{n}u\int_{x_\mathrm{i}}^{x_\mathrm{f}}\frac{\mathrm{d}x}{x-\mathrm{i}\varepsilon} \tag{8}$$

ε being a real infinitesimal shift. As the probability of the under-barrier transition is determined by the imaginary part of S, we find

$$\mathrm{Im}\,S=-\frac{n-1}{n}\pi u\,\mathrm{sgn}(\varepsilon).$$

Evidently, one should choose $\mathrm{sgn}(\varepsilon)=\mathrm{sgn}(n)$, so that the tunnelling action eventually takes the form

$$\mathrm{Im}\,S=-\left|\frac{n-1}{n}\right|\pi u. \tag{9}$$

Recall that the parameter u gives the energy, $U_\mathrm{i}=u^n$, of the initial state, and hence the tunnelling probability depends on the initial point $(x_\mathrm{i}, y_\mathrm{i})$: according to (6), $u=x_\mathrm{i}y_\mathrm{i}$.

The analysis developed can be readily extended for potentials $U(x,y)$ of a more general form, the action integral being determined by the residue of the integrand (4).

In particular, it is straightforward to do this for the potential which is a polynomial in powers of xy. Note, however, that $\operatorname{Im} S$ given by (9) vanishes at $n=1$, when $U(x, y)=xy$. The same holds for all the potentials linear in y, i.e. for

$$U(x, y)=f(x)y+g(x). \tag{10}$$

Note, that the potentials (10) are degenerate in the sense that they give rise to the equation of motion (3a),

$$\dot{x}=-f(x) \tag{11}$$

which does not contain y.

One of the authors (BAM) is indebted to the Institute for Low Temperature Physics and Engineering (Kharkov) for hospitality.

References

[1] Rajaraman R 1982 *Solitons and Instantons* (Amsterdam: North-Holland)

[2] Dunne G, Jackiw R and Trugenberger C 1989 *Ann. Phys., NY* **194** 197
Polychronakos A P 1990 *Ann. Phys., NY* **203** 231

[3] Wilczek F 1990 *Fractional Statistics and Anyon Superconductivity* (Singapore: World Scientific)

[4] Dunne G, Jackiw R and Trugenberger C 1990 *Phys. Rev.* D **41** 661